Synthesis of Decentralized Process Control Structures Using the Concept of Block Relative Gain

The concept of block relative gain (BRG) is introduced to define a measure of interaction for decentralized control structures. This new theoretical development generalizes the original Bristol's relative gain array (RGA) to block pairing of inputs and outputs that are not necessarily single-input single-output pairings. Various properties of BRG are rigorously derived and formulated in a mathematical framework suited to analysis and synthesis. Based on these important properties, a methodology is developed for the systematic generation and selective screening of alternative decentralized control structures. Subsequently, a controller design procedure for the most promising decentralized structures is given. A boiler furnace and a system of heat-integrated reactors are used to illustrate the significance and the utility of the proposed method for industrial processes.

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SCOPE

Control system synthesis consists of two subtasks: selection of measurements and manipulated variables, and deciding on the structure interconnecting the measured and manipulated variables along with the control law governing the interconnections.

Within the last decade several attempts (Govind and Powers, 1982; Umeda and Kuriyama, 1978; Douglas, 1980; Morari et al., 1978; Morari and Stephanopoulos, 1980) have been made to develop mathematical formulations, algorithmic procedures, and systematic strategies to synthesize control structures for integrated complete chemical plants. For details, the reader is referred to the review article by Nishida, et al. (1981). In all these methods, process decomposition (partitioning the process into subsystems) and decentralized control (control of the individual subsystems) has been accepted as the underlying principle. Such an approach not only guides the designer and facilitates the synthesis activity, but in real practice also leads to significant implementational advantages such as improved safety and fault tolerance, reduced communication cost, increased modularity and flexibility to update and expand the control system, and easier process monitoring and on-line controller tuning.

To a large extent, the first subtask of control system synthesis—selection of measurements and manipulated variables—is fairly well understood and is addressed by the existing methods using a combination of heuristic and algorithmic procedures. However the second subtask—selection of the interconnection structure and the control law between the measurements and manipulated variables, which in turn determines the decentralized control system—is far from being resolved. In fact, today there exists no powerful technique to predict the dynamic performance of alternative decentralized control systems and to effectively screen for the best structures. In this paper we address

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these problems and develop the necessary mathematical framework to systematically guide the designer during the synthesis process. Two industrial examples are presented to demonstrate the power of the block relative gain (BRG) concept in developing decentralized control structures.

CONCLUSIONS AND SIGNIFICANCE

A methodology has been presented to quickly inform the designer what alternative decentralized process control structures exist for a set of given manipulated and measured variables and what level of performance can be expected from each. A theoretical framework is developed using a novel concept, block relative gain, which constitutes a dynamic measure of interaction for decentralized control systems. Such an approach allows the designer to exploit a broader class of control structures that are not restricted to the two extremes of complete decentralization (single-input/single-output loops) or complete centralization (full multivariable control systems).

Different block partitioning of input and output sets leads to alternative decentralized control structures, among which the best are selected by a systematic screening procedure that utilizes various important properties of BRG. These properties effectively reduce the combinatorial problems and make the analysis of large-scale systems feasible. Furthermore their appealing features such as scale independence and intimate relationship with the classical relative gain array, among many others, make block relative gain a very promising tool for analysis and synthesis of general structure process control systems.

Introduction

Control system synthesis starts with a given set of measurements, y, and manipulated variables, u. We assume that the plant dynamics is described by an input-output model $y(s) = G(s)u(s)$ in which the transfer function matrix $G(s)$ is considered to be square (i.e., equal number of inputs and controlled outputs).

In decentralized plant control different subsets of outputs, $\{y_i\} \in \{y\}$, are assigned to different subsets of inputs, $\{u_i\} \in \{u\}$, and each such assignment forms a subsystem G_{ii}. In classical feedback terms this implies that output measurements of an individual subsystem will affect the manipulated inputs of that subsystem only via its own control law. Alternative subsystems and thus decentralized control structures can be systematically generated by partitioning $G(s)$ into blocks of different dimensions. For example, 1×1 block partitioning, Figure 1a, gives N subsystems each of which corresponds to a single-input/single-output assignment or a pairing of inputs and outputs that leads to the highest degree of decentralization.

Alternative decentralized control structures arise because of different block-size partitioning of $G(s)$ and because of alternative ways of assigning inputs and outputs to the blocks, Figure 1b. Note that in this type of partitioning subsystems are viewed as aggregates of control loops and not as groups of process units. Thus block partitioning of $G(s)$ may not necessarily correspond to a particular process decomposition, and the resulting decentralized control system does not have to be compatible with any arrangement of subsystems of process unit operations. However this does not preclude the possibility of specifying the process decomposition first and then structuring the decentralized control systems within the boundaries of the individual process subsystems. In some cases this may eliminate the synthesis of undesirable decentralized control structures right from the beginning and reduce the potential combinatorial problems encountered in the block partitioning procedure.

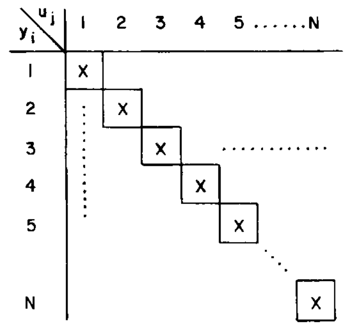

Figure 1a. 1 × 1 block partition of *G*(*s*) · [X]: G_{ii}.

We will discuss the block partitioning procedure in a more algorithmic way after introducing the concept of block relative gain (BRG).

The BRG Concept and Its Properties

The concept of relative gain was originally introduced by Bristol (1966) to provide the designer with a quick assessment of

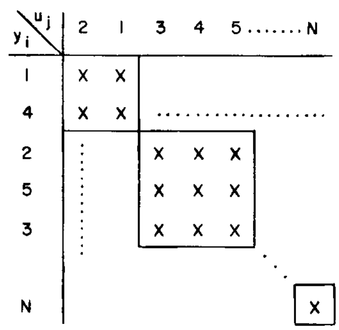

Figure 1b. Partitioning of *G*(*s*) into blocks of different dimensions.

interactions among the control loops of a multivariable system. Using the relative gain interaction measure, the designer tries to select the best controller structure composed of single-input/single-output (SISO) loops, by properly choosing the control pairs (y_i, u_j) among a given set of controlled variables y and a set of manipulated variables u. Because of its many useful properties (McAvoy, 1983) the relative gain method has found wide applications among practicing engineers. However, at the same time, the original development of the relative gain as a scalar and its representation in a single array (i.e., relative gain array—RGA) has unnecessarily limited its applicability exclusively to SISO control loops.

Reformulating and extending the relative gain concept and its properties from a scalar to a matrix (block relative gain) form yields a more powerful synthesis framework that can address a broader class of control problems, such as the synthesis of decentralized control structures that are not restricted to SISO control loops. We now present this new concept of block relative gain.

Definition of BRG

Consider a square $(n \times n)$ transfer function matrix $G(s)$, (in the sequel we will drop s for convenience) partitioned as follows:

$$G = \begin{array}{c} \\ m \\ n-m \end{array} \overset{\leftarrow m \rightarrow \leftarrow n-m \rightarrow}{\left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]} \quad \text{with} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

The plant is to be controlled by a decentralized control structure in which the first m outputs y_1 are interconnected with the first m inputs u_1 and the last $n - m$ outputs y_2 are interconnected with the last $n - m$ inputs u_2. The corresponding feedback configuration is shown in Figure 2 with the controller K and the filter F given by:

$$K = \begin{array}{c} \\ m \\ n-m \end{array} \overset{\leftarrow m \rightarrow \leftarrow n-m \rightarrow}{\left[\begin{array}{c|c} K_1 & 0 \\ \hline 0 & K_2 \end{array}\right]}$$

$$F = \begin{array}{c} \\ m \\ n-m \end{array} \overset{\leftarrow m \rightarrow \leftarrow n-m \rightarrow}{\left[\begin{array}{c|c} F_1 & 0 \\ \hline 0 & F_2 \end{array}\right]} \tag{2}$$

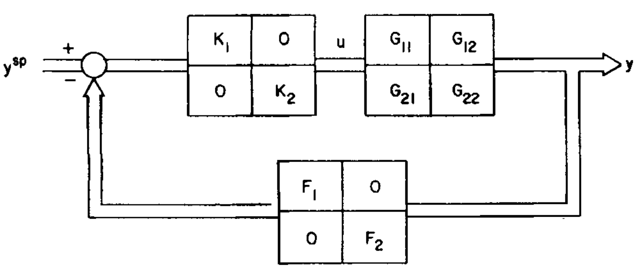

Figure 2. Decentralized feedback system.

The following relations hold:

$$y = Gu \tag{3}$$

$$u = G^{-1}y \quad (\text{assuming } G^{-1} \text{ exists}) \tag{4}$$

Then

$$\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{u_2=0 \\ F=0}} = G_{11} \tag{5}$$

$$\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}} = ([G^{-1}]_{11})^{-1}$$

$$= G_{11} - G_{12}G_{22}^{-1}G_{21} \quad (\text{if } G_{22} \text{ is nonsingular}) \tag{6}$$

where $[G^{-1}]_{11}$ is the first $m \times m$ block of G^{-1}:

$$G^{-1} = \begin{bmatrix} [G^{-1}]_{11} & [G^{-1}]_{12} \\ [G^{-1}]_{21} & [G^{-1}]_{22} \end{bmatrix} \tag{7}$$

According to Eq. 5 G_{11} denotes the block gain between y_1 and u_1 when all the loops are open, i.e., $F = 0$. Similarly, $([G^{-1}]_{11})^{-1}$ is the block gain between y_1 and u_1 when the first m loops are open, i.e., $F_1 = 0$, and the last $n - m$ loops are closed, i.e., $F_2 = I$, and under perfect control, i.e., $y_2 = 0$. In these definitions transfer function operators are loosely referred to as block gains in order to adhere to the original notation of Bristol without creating additional unnecessary terminology.

We can now define the m-dimensional block relative gain (left and right) as:

$$BRG_\ell = \left[\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{u_2=0 \\ F=0}}\right] \cdot \left[\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}}\right]^{-1} = G_{11} \cdot [G^{-1}]_{11} \tag{8}$$

$$BRG_r = \left[\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}}\right]^{-1} \cdot \left[\left.\frac{\partial y_1}{\partial u_1}\right|_{\substack{u_2=0 \\ F=0}}\right] = [G^{-1}]_{11} \cdot G_{11} \tag{9}$$

Note that in the case of one-dimensional BRG, left and right BRG's become identical (since G_{11} is scalar) and reduce to the classical Bristol's relative gain. In the case of n-dimensional BRG, $BRG_\ell = BRG_r = I$ (identity matrix).

Closed-loop performance and BRG

The significance of BRG in relation to the closed-loop performance will be derived from a study of the following three cases:

(i) $F_1 = 0, F_2 = 0$ (no feedback)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} y_1^{sp} \\ y_2^{sp} \end{bmatrix} \tag{10}$$

$$\left.\frac{\partial y_1}{\partial y_1^{sp}}\right|_{\substack{y_2^{sp}=0 \\ F=0}} = G_{11}K_1 \tag{11}$$

(ii) $F_1 = 0, F_2 = I$ (feedback of the last $n - m$ outputs to the last $n - m$ inputs)

$$\begin{bmatrix} y_1^{sp} \\ y_2^{sp} \end{bmatrix} = \begin{bmatrix} K_1^{-1}[G^{-1}]_{11} & K_1^{-1}[G^{-1}]_{12} \\ K_2^{-1}[G^{-1}]_{21} & I + K_2^{-1}[G^{-1}]_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{12}$$

$$\left. \frac{\partial y_1^{sp}}{\partial y_l} \right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}} = K_1^{-1}[G^{-1}]_{11} \tag{13}$$

(iii) $F_1 = I, F_2 = I$ (feedback of the first m [last $n - m$] outputs to the first m [last $n - m$] inputs, respectively)

$$\begin{bmatrix} y_1^{sp} \\ y_2^{sp} \end{bmatrix} = \begin{bmatrix} I + K_1^{-1}[G^{-1}]_{11} & K_1^{-1}[G^{-1}]_{12} \\ K_2^{-1}[G^{-1}]_{21} & I + K_2^{-1}[G^{-1}]_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{14}$$

$$\left. \frac{\partial y_1^{sp}}{\partial y_l} \right|_{\substack{y_2=0 \\ F=I}} = I + K_1^{-1}[G^{-1}]_{11} \tag{15}$$

Using these relations, we can derive the following:

$$\begin{aligned} BRG_\ell &= G_{11} \cdot [G^{-1}]_{11} = G_{11}K_1 \cdot K_1^{-1}[G^{-1}]_{11} \\ &= \left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{y_2^{sp}=0 \\ F=0}} \cdot \left. \frac{\partial y_1^{sp}}{\partial y_l} \right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}} \end{aligned} \tag{16}$$

$$\left. \frac{\partial y_1^{sp}}{\partial y_l} \right|_{\substack{y_2=0 \\ F=I}} = I + K_1^{-1}[G^{-1}]_{11} = I + \left. \frac{\partial y_1^{sp}}{\partial y_l} \right|_{\substack{y_2=0 \\ F_1=0 \\ F_2=I}} \tag{17}$$

Using Eq. 16, Eq. 17 becomes:

$$\left. \frac{\partial y_l}{\partial y_1^{sp}} \right|_{\substack{y_2=0 \\ F=I}} = \left[I + \left(BRG_\ell^{-1} \cdot \left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{y_2^{sp}=0 \\ F=0}} \right)^{-1} \right]^{-1} \tag{18}$$

Thus BRG_ℓ^{-1} is the factor by which the open-loop gain $G_{11}K_1$ must be premultiplied so that the effect of the other loops is taken into account in the closed-loop response of y_1 to its set points y_1^{sp}. When $BRG_\ell = I$, Eq. 18 gives

$$\begin{aligned} \left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{y_2=0 \\ F=I}} &= \left[I + \left[\left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{y_2^{sp}=0 \\ F=0}} \right]^{-1} \right]^{-1} \\ &= [I + (G_{11}K_1)^{-1}]^{-1} \end{aligned} \tag{19}$$

However note that $[I + (G_{11}K_1)^{-1}]^{-1}$ is the closed-loop transfer function matrix between y_1 and y_1^{sp} when there exist no other loops. Thus we have the following result: If $BRG_\ell = I$ then

$$\left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{y_2=0 \\ F=I}} = \left. \frac{\partial y_1}{\partial y_1^{sp}} \right|_{\substack{\nexists (y_2, u_2) \\ F_1=I}} \tag{20}$$

Equations 18–20 provide the answer to the prolonged question: What is the significance of the relative gain for the performance of the closed-loop system? The answer is: The closed-loop performance of the $m \times m$ block under consideration, when the other $n - m$ outputs are under perfect control, is a continuous function of BRG_ℓ as dictated by Eq. 18. When $BRG_\ell = I$, which implies $BRG_r = I$, the closed-loop performance of the $m \times m$ block is as if this block were isolated from the rest of the plant and operating under the influence only of its own control law. This makes it clear what kind of information one should expect from BRG and in what sense it can be considered as a measure of interaction.

Finally one should note that while defining the block relative gain and deriving its relation to the closed-loop performance we have accepted the usual assumption of perfect control for the plant outputs y_2. This assumption always holds at zero frequency (i.e., at steady state) by the use of integral control action. However it may not hold at all the frequencies when nonminimum phase and/or strictly proper blocks are present. For such cases the assumption of perfect control over the whole frequency range can be relaxed, and as a result the equations derived so far are slightly modified (for details see Manousiouthakis, 1985).

Our conclusion at this point is that if one wished to investigate interactions over the whole frequency range, BRG given by Eqs. 8 and 9 is a dependable frequency-dependent interaction measure and need not be modified if there are no right half-plane transmission zeros in the complementary subsystem, which is supposed to work under perfect control. On the other hand, when RHP zeros exist, one either evaluates Eqs. 8 and 9 at steady state only or, if interested in all the frequencies, uses an appropriately modified BRG as given in Manousiouthakis (1985). In our physical examples we will address systems without RHP zeros for which BRG given by Eqs. 8 and 9 can be safely used at all the frequencies.

Properties of BRG

BRG, as it is defined above, is related to the first m outputs and m inputs of the plant. As a result, it will depend on how the n outputs and n inputs are ordered in $G(s)$. Since the number of all possible rearrangements of n objects is $n!$, n outputs and n inputs can be ordered in $(n!) \cdot (n!)$ possible combinations. Calculating an m-dimensional BRG for each such combination would result to a total of $(n!)^2$ BRG computations (see Eqs. 8 and 9), which would be an enormous task for large n's. In order to resolve this combinatorial problem, a mathematical statement of the problem is attempted and a theorem is presented.

The set of transfer functions that corresponds to all possible rearrangements of inputs and outputs can be described by:

$$\mathcal{G} = \{G' \in R^{n \times n}(s) / G' = P_1 G P_2\} \tag{21}$$

where G is the transfer function matrix corresponding to a particular ordering of inputs and outputs. P_1 and P_2 are real square permutation matrices with zero entries everywhere except at one element in each row and column which assumes the value 1.

Let G be an $n \times n$ matrix partitioned as follows:

$$G = \begin{array}{c} \scriptstyle m \\ \scriptstyle n-m \end{array} \overset{\begin{array}{cc} \scriptstyle m & \scriptstyle n-m \end{array}}{\left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array} \right]} \tag{22}$$

Consider a permutation $G' = P_1 G P_2$ where P_1 and P_2 have the following special form:

$$G' = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{12} \end{bmatrix} \cdot G \cdot \begin{bmatrix} P_{21} & 0 \\ 0 & P_{22} \end{bmatrix} \tag{23}$$

Here P_{11} permutes the first m outputs and P_{12} permutes the last $n - m$ outputs. Similarly, P_{21} permutes the first m inputs and P_{22} permutes the last $n - m$ inputs. Thus the rearrangements of inputs and outputs are confined within the partitioned blocks of G with no permutation across the blocks.

The following theorem reduces the number of BRG's to be evaluated to a manageable size.

Theorem 1. Let G and G' be given by Eqs. 22 and 23, respectively. Then the m-dimensional BRG of G', denoted by $(BRG)'$ is given by

$$(BRG)'_\ell = P_{11} \cdot BRG_\ell \cdot P_{11}{}^T \tag{24}$$

$$(BRG)'_r = P_{21}{}^T \cdot BRG_r \cdot P_{21} \tag{25}$$

where BRG_ℓ and BRG_r are those of G, and T denotes the transpose.

Proof. See Manousiouthakis (1985).

Theorem 1 proves that BRG_ℓ (BRG_r) is not affected at all by the ordering of the last $n - m$ inputs and $n - m$ outputs and the ordering of the first m inputs (outputs). Furthermore, for all G''s $\in \mathcal{G}$ that contain the same first m inputs and m outputs but in different arrangements, corresponding $(BRG)'$'s turn out to be trivial rearrangements of each other as shown by Eqs. 24 and 25. Consequently, they can be considered equivalent. Thus for an m-dimensional subsystem containing a unique group of inputs and outputs only one of the equivalent BRG's needs to be examined. This means that for an n-dimensional system the number of calculations for an m-dimensional BRG drops from $(n!)^2$ to $\binom{n}{m} \cdot \binom{n}{m}$, which is a significant reduction for large systems.

Theorem 2. Let G by an $n \times n$ matrix partitioned as in theorem 1. Consider the scaled matrix $G^s = S_1 G S_2$, where S_1 and S_2 are real, diagonal input and output scaling matrices, respectively, and are partitioned as follows:

$$S_1 = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix} \quad S_2 = \begin{bmatrix} S_{21} & 0 \\ 0 & S_{22} \end{bmatrix}$$

with $S_{11}, S_{21} \in R^{m \times m}$ and $S_{12}, S_{22} \in R^{(n-m)\times(n-m)}$. Then the m-dimensional BRG of G^s, denoted by $(BRG)^s$ is given by

$$(BRG)^s_\ell = S_{11} \cdot BRG_\ell \cdot S_{11}^{-1} \tag{26}$$

$$(BRG)^s_r = S_{21}^{-1} \cdot BRG_r \cdot S_{21} \tag{27}$$

where BRG_ℓ and BRG_r are the block relative gains of G.

Proof. See Manousiouthakis (1985).

Theorem 2 proves the following: $BRG_\ell(BRG_r)$ does not depend on the scaling of the last $n - m$ inputs and outputs and the first m inputs (outputs), but it does depend on the scaling of the first m outputs (inputs). However if the first m outputs (inputs) are all scaled the same way, i.e., S_{11} (S_{21}) is a scalar multiple of the identity matrix, then $BRG_\ell(BRG_r)$ is not affected at all.

Theorem 3. Consider G''s $\in \mathcal{G}$ with P_1, P_2 having the special form of theorem 1. Then the diagonal elements of $(BRG)'_\ell$, $(BRG)'_r$ remain the same for all G''s that belong to this subset of $\mathcal{G}$.

Proof. See Appendix.

According to theorem 3, the diagonal elements of $(BRG)'$ are well-defined, i.e., they remain on the diagonal but not necessarily at the same locations when G' is trivially rearranged. This implies that for all the BRG's corresponding to a particular group of m inputs and outputs the designer will need to examine only m diagonal terms.

The next theorem introduces additional significance to the diagonal elements of BRG and establishes them as a basic aid in the decentralized control structure selection process.

Theorem 4. Let BRG_ℓ and BRG_r be the m-dimensional block relative gains corresponding to the first m outputs and inputs of a given system G. Their diagonal elements are independent of scaling and can be expressed in terms of the one-dimensional BRG's that correspond to these m outputs and m inputs (i.e., in terms of the corresponding RGA elements). Specifically

$$(BRG)_\ell(i, i) = \sum_{j=1}^{m} \mathrm{RGA}(i, j) \tag{28}$$

$$(BRG)_r(i, i) = \sum_{j=1}^{m} \mathrm{RGA}(j, i) \tag{29}$$

Proof. See Appendix.

It is important to note that the well-known property of the relative gain array, that elements of each row and each column add to 1, is a direct consequence of theorem 4 (applied for $m = n$) and the fact that the n-dimensional BRG is the identity matrix.

The next theorem establishes some important properties for the eigenvalues of BRG, which will be useful during the synthesis of decentralized control systems.

Theorem 5. Let BRG_ℓ and BRG_r be the m-dimensional block relative gains corresponding to the first m outputs and inputs of a given system G. Then their eigenvalues $\lambda_i(BRG)_\ell$, $\lambda_i(BRG)_r$; $i = 1, \ldots, m$:

- Are independent of scaling
- Remain the same for all G''s that belong to the subset considered in theorems 1 and 3
- Are the same for both BRG_ℓ and BRG_r, that is $\lambda_i(BRG)_\ell = \lambda_i(BRG)_r$; $i = 1, \ldots, m$

Proof. See Appendix.

Block Partitioning and Control Structure Selection Methodology

The aim of this section is to provide an acceptable block partitioning of the plant matrix $G(s)$. Such a task is considered to be accomplished if all the BRG's of different dimensions corresponding to the diagonal blocks of different dimensions, $G_{ii}(s)$'s, are close to an identity matrix.

Closeness and viability of BRG's

According to its definition, an m-dimensional BRG belongs to the algebra of $m \times m$ matrices whose elements are rational functions of s. It can be shown that topological properties of the elements of this algebra are related only to the behavior of these elements on the imaginary axis $s = j\omega$. Thus in order to assess the closeness of BRG to the identity matrix, one need only evaluate m-dimensional $BRG|s = j\omega$ as a function of the frequency ω. We next quantify this closeness and define the set of viable BRG's.

Let $B(1, \epsilon)$ denote a neighborhood in the complex plane with center at (1, 0) and radius $\epsilon = \epsilon(\omega)$. Then we say that a BRG is viable (i.e., close to identity) if its diagonal elements and eigenvalues belong to neighborhoods $B(1, \epsilon_1)$ and $B(1, \epsilon_2)$, respectively, for all frequencies ω. Note that since the diagonal elements and eigenvalues of BRG possess various important properties (see theorems 4 and 5), they will be heavily utilized during the selection procedure.

Acceptable partitioning

A set of viable BRG's is said to yield an acceptable block partitioning of the plant if the output (input) sets of these BRG's are mutually exclusive and their union comprises the output (input) set of the whole plant. Acceptability simply implies that control structures for the partitioned blocks are decentralized (i.e., they do not have common inputs and outputs) and there are enough decentralized control structures to control all the outputs.

Having defined closeness, viability, and acceptability, one can start the selection process.

Selection process

First consider the highest degree of decentralization—i.e., 1×1 block partitioning of G—that would yield a total of N SISO assignments (or pairings). For this, all the one-dimensional BRG's (or the familiar RGA) are first evaluated at $s = 0$. Note that in this case $[(BRG)_\ell = (BRG)_r]$ and one-dimensional BRG's are by definition the diagonal elements and the eigenvalues. It is then easy to identify the viable one-dimensional BRG's for $s = 0$ since they are all real. Among the viable ones, those which establish a 1–1 correspondence between the plant's inputs and outputs are selected. If such alternatives do not exist, then there is no acceptable partitioning using 1×1 blocks only. In that case assignment is not complete and one proceeds with two-dimensional BRG's. In case there exists an acceptable 1×1 block partitioning for $s = 0$ but viability and/or acceptance are violated at frequencies other than $\omega = 0$, the study of two-dimensional BRG's is again necessary. Otherwise the procedure can conclude at this point with the resulting 1×1 block partitioning and the corresponding control structures.

The next step in the process is the study of two-dimensional BRG's. We first study only BRG_ℓ's. This is because BRG_ℓ's carry much more physical significance than BRG_r's since they are intimately related to the closed-loop response (see Eq. 18 and the subsequent discussion). The essence of the screening process is that one first screens out BRG_ℓ's whose diagonal elements are not close to 1. Among the remaining BRG_ℓ's, only those with eigenvalues close to 1 are retained since these are the BRG_ℓ's that are close to the identity matrix. Since the eigenvalues of BRG_r and BRG_ℓ are the same (see theorem 5) a detailed study of BRG_r's is deemed unnecessary in case the eigenvalues of BRG_ℓ are close to 1. If this is not the case, the diagonal elements of BRG_r's should be calculated from the RGA (based on theorem 4) and their closeness to one should be examined as a final screening criteria. The following procedure systematizes the screening process which starts with $s = 0$.

According to theorem 4, the diagonal terms of all the two-dimensional BRG_ℓ's are the elements of the column vectors that result from every possible addition of two columns of the RGA. Thus if one of these column vectors has $p > 2$ elements within $B(1, \epsilon_1)$ this implies that there exist only $p!/[2!(p-2)!]$ two-dimensional BRG_ℓ's that should be further considered. Among these BRG_ℓ's those with eigenvalues outside $B(1, \epsilon_2)$ are rejected. The remaining BRG_ℓ's are the two-dimensional viable BRG_ℓ's for $s = 0$ and for one of the column vectors discussed above. The screening process is repeated for all the possible column vectors and for all frequencies other than $\omega = 0$ and ultimately gives all viable two-dimensional BRG_ℓ's.

Searching for an acceptable partitioning over the sets of both two- and one-dimensional viable BRG_ℓ's is the next step. If one is found, the procedure concludes; otherwise it continues with the study of BRG_ℓ's of higher dimension in the same manner until a solution is achieved. The process is guaranteed to conclude since in the worst case it will lead to a centralized full control structure that corresponds to an n-dimensional BRG.

At this point it should be mentioned that $\epsilon_1(\omega)$, $\epsilon_2(\omega)$ are free parameters through which the designer can affect the screening process and establish what an acceptable degree of interaction is.

Controller Design

Once the partitioning has been selected the controller design task must be undertaken. The underlying structure is the model

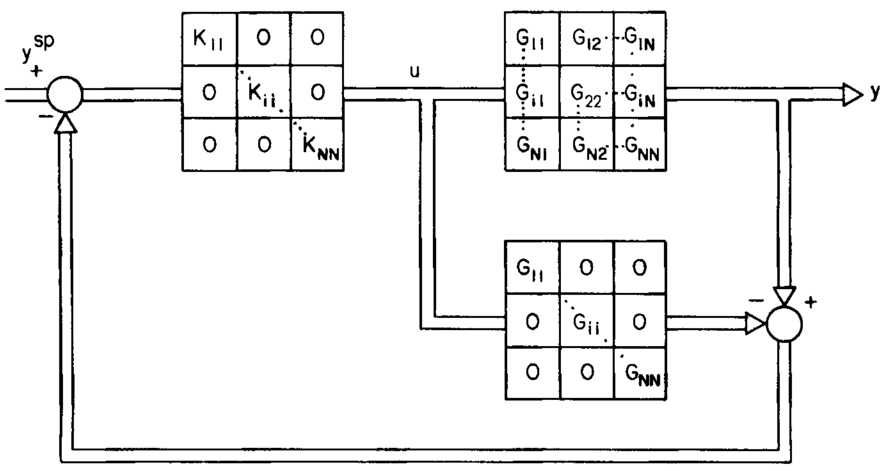

Figure 3. Decentralized model reference scheme.

reference scheme as applied to a collection of subsystems [i.e., partitioned blocks of $G(s)$] and is depicted in Figure 3. The equivalence between this scheme and the decentralized feedback scheme of Figure 2 constitutes the foundation of the design method.

The controller design can proceed independently for each block G_{ii} by employing the design procedure for stable plants presented in Manousiouthakis and Arkun (1984). For each block in Figure 3, the corresponding transfer function matrix G_{ii} can be factorized as $G_{ii} = \hat{G}_{ii}\tilde{G}_{ii}$ where $\hat{G}_{ii}$, $\tilde{G}_{ii}$ are the noninvertible and invertible parts of G_{ii}, respectively. Then the ith subsystem model reference controller is $G_{ii} = \tilde{G}_{ii}^{-1} \cdot M_i$ where M_i is selected stable and minimum-phase.

The classical feedback controller K_{ii} corresponding to the ith subsystem can be obtained, based on the equivalence of the two schemes, from the model reference controller G_{ii} as follows:

$$K_{ii} = [G_{ii}^{-1} - G_{ii}]^{-1} = \tilde{G}_{ii}^{-1}(M_i^{-1} - \hat{G}_{ii})^{-1}.$$

The feedback controllers K_{ii} are thus parameterized in terms of M_i and contain integral action when $\hat{G}_{ii}(o) \cdot M_i(o) = I$. In the case that the blocks G_{ii} are minimum-phase, as in the examples studied, M_i's can be chosen so that $\hat{G}_{ii}M_i$'s have the form

$$\hat{G}_{ii}M_i = \text{diag}\left(\frac{1}{(\alpha s + 1)^{k_j}}\right)$$

where $\alpha > 0$ and k_j is an integer; $i = 1, \ldots, N$; $j = j(i) = 1, \ldots, n_i$. When BRG_ℓ deviates significantly from I at high frequencies then the tuning parameter α should be selected large enough so that closed-loop stability is guaranteed (Manousiouthakis and Arkun, 1985).

Examples

Two industrial examples are presented to demonstrate the power of the BRG concept in developing decentralized control structures. Dynamic simulations are also given to verify the prediction of the BRG analysis.

Boiler furnace

The first system under consideration is a furnace operating with four burners and four heating coils, as shown in Figure 4. The coil temperatures are to be controlled by manipulating the gas flow rates of the burners. The plant transfer function matrix can be found in Rosenbrock (1974).

The selection procedure starts by specifying the radii $e_1(\omega)$ and $e_2(\omega)$ for the diagonal elements and the eigenvalues of BRG, respectively. For this example they are selected to be $\epsilon_1(\omega) = 0.15$ and $\epsilon_2(\omega) = 0.2$ for all frequencies. Inspection of the one-dimensional BRG's$|_{s=0}$, shown below in the RGA form, makes apparent that no viable ones exist for this choice of ϵ_1 and ϵ_2 and the two-dimensional BRG's have to be examined.

One-dimensional *BRG*'s

y_i	u_j: 1	2	3	4
1	1.748	−0.686	−0.096	0.034
2	−0.727	1.874	−0.092	−0.055
3	−0.055	−0.092	1.874	−0.727
4	0.034	−0.096	−0.686	1.748

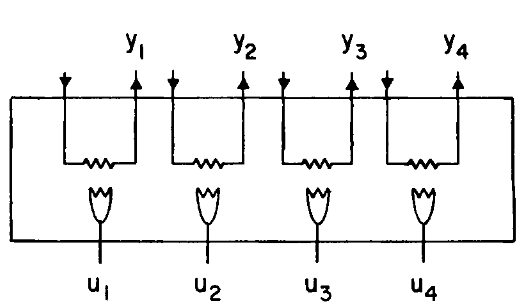

Figure 4. Sketch of boiler furnace.

The diagonal elements of the two-dimensional BRG_ℓ's$|_{s=0}$ are available when all possible additions of two columns of the above RGA are performed. The resulting column vectors are shown below.

y_i	u_j: 1 + 2	1 + 3	1 + 4	2 + 3	2 + 4	3 + 4
1	**1.062**	1.652	1.782	−0.782	−0.652	−0.062
2	**1.147**	−0.819	−0.782	1.782	1.819	−0.147
3	−0.147	1.819	−0.782	1.782	−0.819	**1.147**
4	−0.062	−0.652	1.782	−0.782	1.652	**1.062**

The only promising elements are those marked in columns 1 + 2, 3 + 4. The corresponding two-dimensional BRG$_\ell$'s are given below.

y_i	u_j: 1	2
1	1.062	0.101
2	0.069	1.147

y_i	u_j: 3	4
3	1.062	0.101
4	0.069	1.147

Figure 5 shows that the diagonal elements belong to $B(1, \epsilon_1)$ for all frequencies.

Next the eigenvalues are examined. Inspection of Figure 6 verifies that these eigenvalues belong to $B(1, \epsilon_2)$ for all frequencies and thus these BRG_ℓ's are viable. Note that because of the physical symmetry (Figure 4), the eigenvalues of the two *BRG*'s are identical. Since the output and input sets are also mutually exclusive, these *BRG*'s complement each other and yield an acceptable block partitioning: Block 1: $\{y_1, y_2; u_1, u_2\}$ and Block 2: $\{y_3, y_4; u_3, u_4\}$.

Working in the same way one can verify that no viable three-dimensional BRG's exist for $\epsilon_1 = 0.15$ and $\epsilon_2 = 0.2$. By relaxing the constraints $\epsilon_1(\omega)$, $\epsilon_2(\omega)$ one can increase the number of alternatives although these new structures will be less desirable. The most promising one-dimensional BRG's are for $\{y_1; u_1\}$, $\{y_2; u_2\}$, $\{y_3; u_3\}$, $\{y_4; u_4\}$, which can also form an acceptable partitioning if ϵ_1 and ϵ_2 are increased to 0.874 (see RGA). The most promising three-dimensional BRG_ℓ's correspond to $\{y_1, y_2, y_3; u_1, u_2, u_3\}$, and $\{y_2, y_3, y_4; u_2, u_3, u_4\}$ and are given below:

yi	u_j: 1	2	3
1	0.966	0.032	0.242
2	−0.059	1.056	0.424
3	−0.102	0.095	1.727

y_i	u_j: 2	3	4
2	1.727	0.095	−0.102
3	0.4239	1.056	−0.059
4	0.242	0.032	0.966

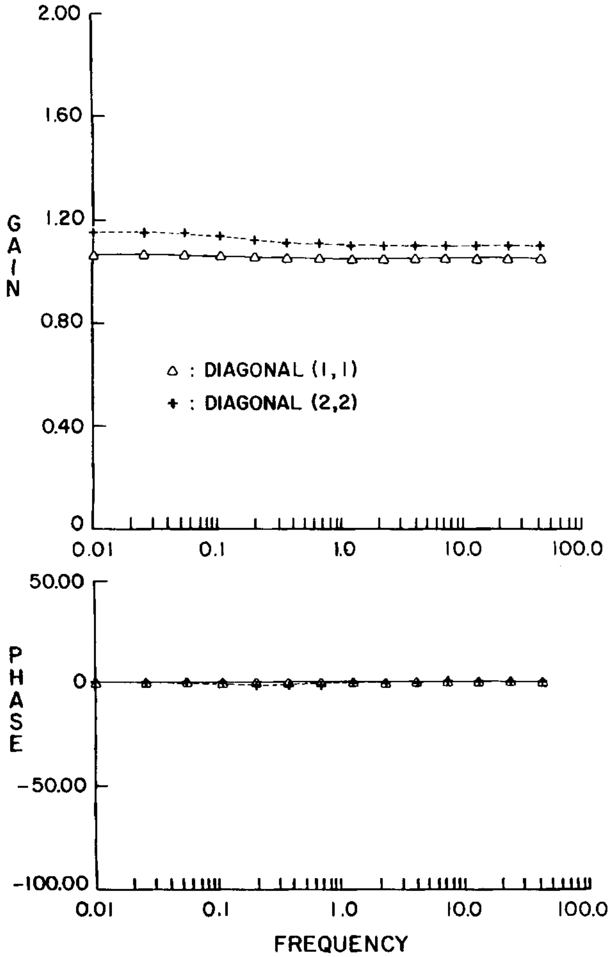

Figure 5a. Diagonal elements of the two-dimensional BRG $\{y_1, y_2; u_1, u_2\}$ as a function of frequency.

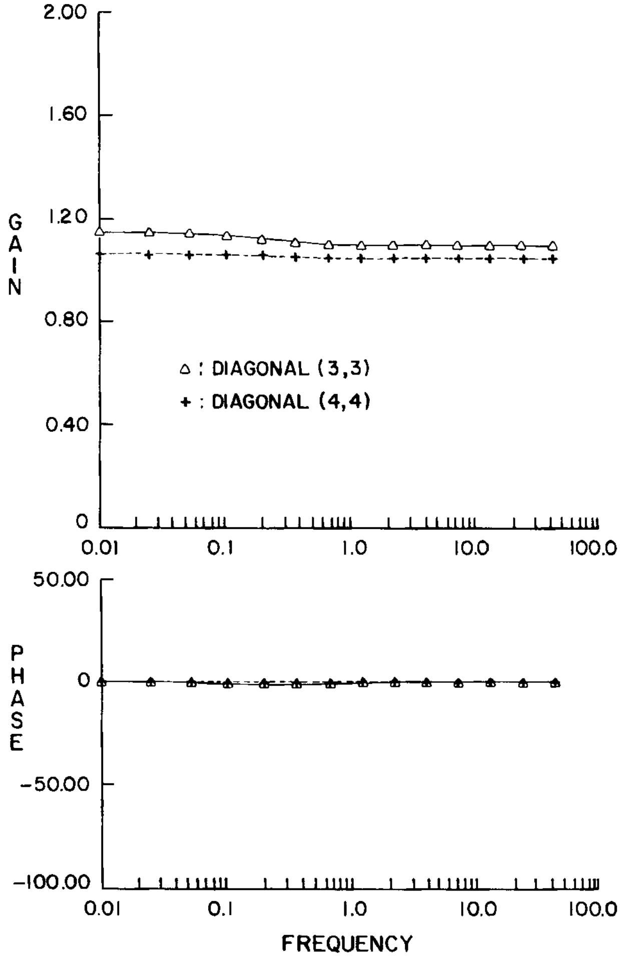

Figure 5b. Diagonal elements of the two-dimensional BRG $\{y_3, y_4; u_3, u_4\}$ as a function of frequency.

Because of the physical symmetry (Figure 4), the two BRG's are similar (i.e., one can be obtained from the other by simple rearrangement of the elements). The above three-dimensional blocks when combined with $\{y_4; u_4\}$ and $\{y_1; u_1\}$, respectively, form acceptable partitionings if ϵ_1 and ϵ_2 are increased to 0.748.

In summary, the following alternatives are promising:

1. $\{y_1, y_2; u_1, u_2\}$ and $\{y_3, y_4; u_3, u_4\}$ acceptable if $\epsilon_1 = 0.15$, $\epsilon_2 = 0.2$.
2. $\{y_1, u_1\}$, $\{y_2, u_2\}$, $\{y_3, u_3\}$, $\{y_4, u_4\}$, acceptable if $\epsilon_1 = 0.874$, $\epsilon_2 = 0.874$.
3. $\{y_1, y_2, y_3; u_1, u_2, u_3\}$ and $\{y_4, u_4\}$ acceptable if $\epsilon_1 = 0.748$, $\epsilon_2 = 0.748$.
4. $\{y_2, y_3, y_4; u_2, u_3, u_4\}$ and $\{y_1, u_1\}$ acceptable if $\epsilon_1 = 0.748$, $\epsilon_2 = 0.748$.

Because of larger ϵ_i values the last three alternatives are less desirable than the first one.

To illustrate the validity of the above results, decentralized controller design as discussed earlier will be attempted for the promising control structures and simulations will be carried out. For all the alternatives, G_{ii}'s are minimum-phase and stable and $\tilde{G}_{ii}M_i$ has the form $\tilde{G}_{ii}M_i$ = diag $[1/(0.2s + 1)]$. Simulations of the closed-loop system, Figure 7, demonstrate that the partitioning in two two-dimensional blocks, which is suggested by the above BRG analysis, is better than all other acceptable partitionings. In fact it is even better than the partitioning $\{y_1, y_2, y_3; u_1, u_2, u_3\}$, $\{y_4; u_4\}$, which has a more centralized control action. Note that the dynamic performances of all the selected decentralized structures are acceptable and the predictions of the BRG analysis are completely verified.

One should also notice that the SISO partitioning suggested here is the same as that arrived at by Rosenbrock (1974) based on physical arguments. However the 2-2 structure $\{y_1, y_2; u_1, u_2\}$, $\{y_3, y_4; u_3, u_4\}$, which is identified by the BRG analysis to be the best, cannot readily be recognized as the best alternative based on physical intuition alone.

Interestingly enough, for this example the above decentralized control structures also correspond to particular process decompositions. For example, SISO decentralized controllers correspond to decomposition of the boiler furnace into four similar compartments and the 2-2 decentralized structure corresponds to decomposition of the furnace into two symmetric compartments.

Heat-integrated reactors

The second example physical plant consists of two continuous stirred-tank reactors integrated with two heat exchangers as shown in Figure 8. A preliminary selection of the outputs and inputs of the process has been done by Morari et al. (1983). The present study investigates the potential of decentralized control.

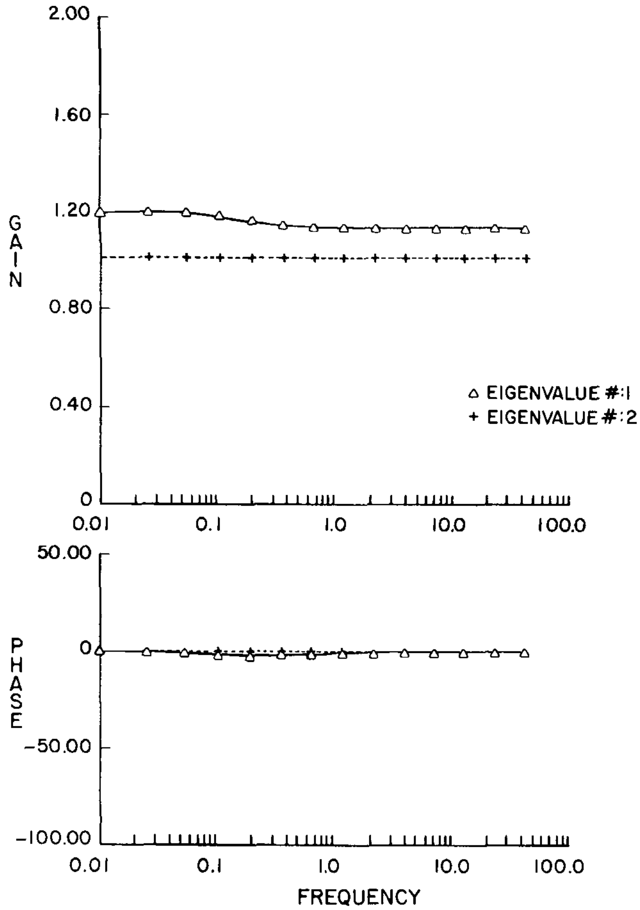

Figure 6. Eigenvalues of the two-dimensional BRG $\{y_1, y_2; u_1, u_2\}$ as a function of frequency.

Four intermediate process stream temperatures (T_5, T_6, T_7, T_8) are to be controlled by manipulating the bypass valves (v_1, v_2) in the two heat exchangers and the valves (v_3, v_4) on the two cooling coils of the reactors.

$G(s)$ is computed from the data given in Morari et al. (1983).

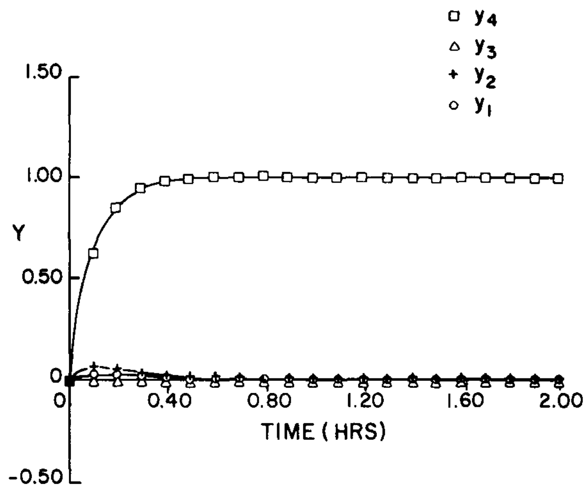

Figure 7b. Set point responses for the 2-2 partitioning: $\{y_1, y_2; u_1, u_2\}$, $\{y_3, y_4; u_3, u_4\}$.

The selection procedure begins by calculating all the one-dimensional BRG's at $s = 0$ and setting the radii ϵ_1, ϵ_2 to 0.1. These are given below:

One-dimensional BRG's at $s = 0$

	u_j			
y_i	1	2	3	4
1	0.570	0.470	−0.040	0.0
2	0.030	0.070	0.900	0.0
3	0.475	0.445	0.171	−0.091
4	−0.075	0.015	−0.031	1.091

According to the above results, one-dimensional BRG's that correspond to $\{y_2, u_3\}$, $\{y_4, u_4\}$ lie within $B(1, 0.1)$ at $s = 0$. Figure 9 shows that these BRG's lie within $B(1, 0.1)$ for all frequencies. Thus they are viable but they are not acceptable, since partitioning is not complete. One then has to proceed with two-dimensional BRG_ℓ's.

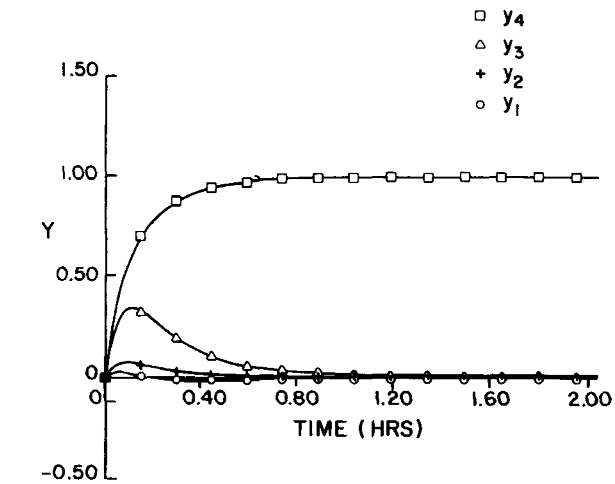

Figure 7a. Set point responses for the 1-1-1-1 partitioning: $\{y_1, u_1\}$, $\{y_2, u_2\}$, $\{y_3, u_3\}$, $\{y_4, u_4\}$.

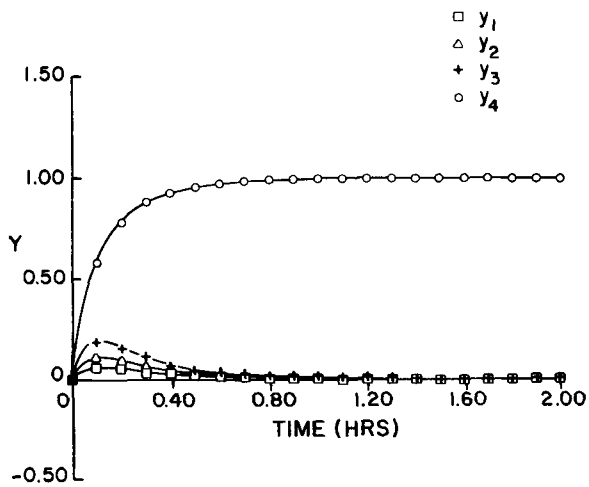

Figure 7c. Set point responses for the 3-1 partitioning: $\{y_1, y_2, y_3; u_1, u_2, u_3\}$, $\{y_4, u_4\}$.

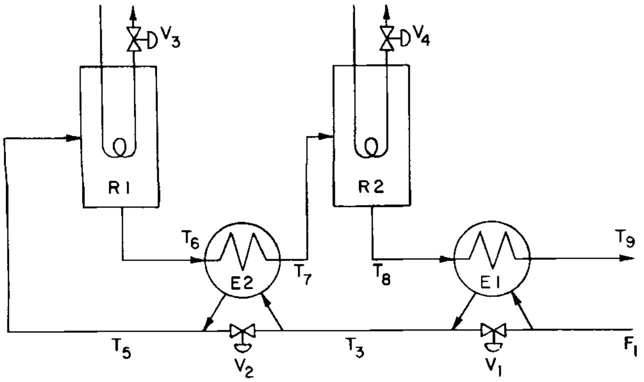

Figure 8. Heat-integrated reactors.

The column vectors containing the diagonal elements of all the two-dimensional BRG_ℓ's are given below:

y_i \ u_j	1 + 2	1 + 3	1 + 4	2 + 3	2 + 4	3 + 4
1	1.039	0.531	0.571	0.429	0.469	−0.039
2	0.100	0.929	0.029	0.971	0.071	0.900
3	0.921	0.646	0.384	0.616	0.354	0.079
4	−0.06	−0.106	1.016	−0.016	1.106	1.06

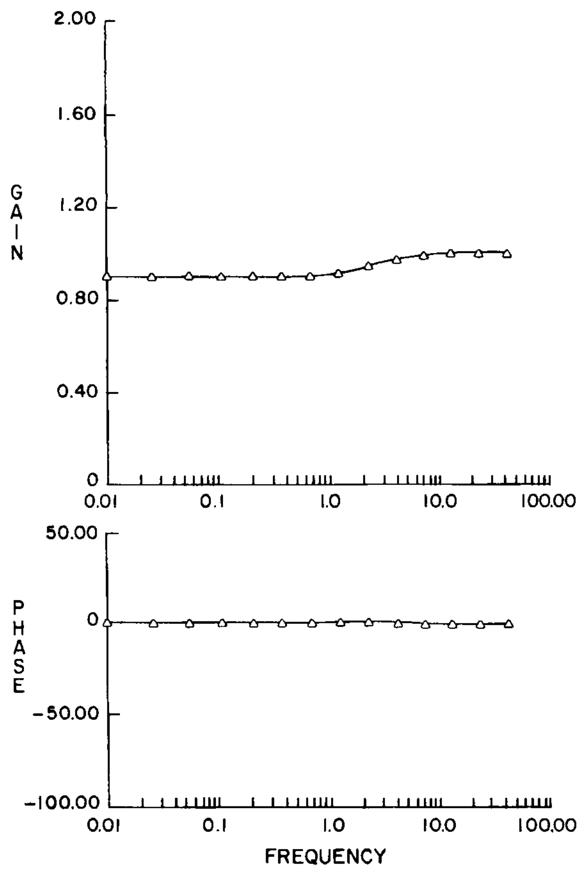

Figure 9a. One-dimensional BRG of $\{y_2, u_3\}$.

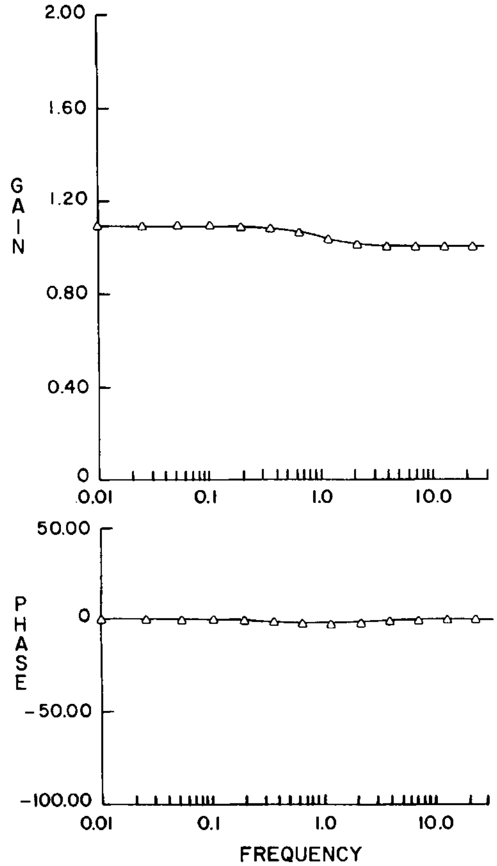

Figure 9b. One-dimensional BRG of $\{y_4, u_4\}$.

Since in a column vector there must be at least two elements within $B(1, 0.1)$ in order to make up a viable two-dimensional BRG_ℓ, the only BRG_ℓ's that qualify for further screening are those of $\{y_1, y_3; u_1, u_2\}$ and $\{y_2, y_4; u_3, u_4\}$:

Two-dimensional Viable BRG_ℓ's at $s = 0$

y_i	u_j = 1	2	y_i	u_j = 3	4
1	1.039	0.048	2	0.9	0.001
3	0.069	0.921	4	0.316	1.06

Since their diagonal elements and eigenvalues can be shown to lie within $B(1, 0.1)$ for all frequencies, these BRG's are viable. The partitioning process could be stopped here since acceptable block partitions can be developed from the viable one- and two-dimensional BRG's generated up to now:

Acceptable Block Partitionings.

Alternative 1. Block 1: $\{y_4, u_4\}$; Block 2: $\{y_2, u_3\}$; Block 3: $\{y_1, y_3; u_1, u_2\}$.

Alternative 2. Block 1: $\{y_1, y_3; u_1, u_2\}$; Block 2: $\{y_2, y_4; u_3, u_4\}$.

The partitioning process can be continued with an analysis of three-dimensional BRG's to create additional alternatives. The column vectors containing the diagonal elements of all the three-dimensional BRG_ℓ's at $s = 0$ are given below [in a particular column there must be at least three elements within $B(1, \epsilon_1)$]:

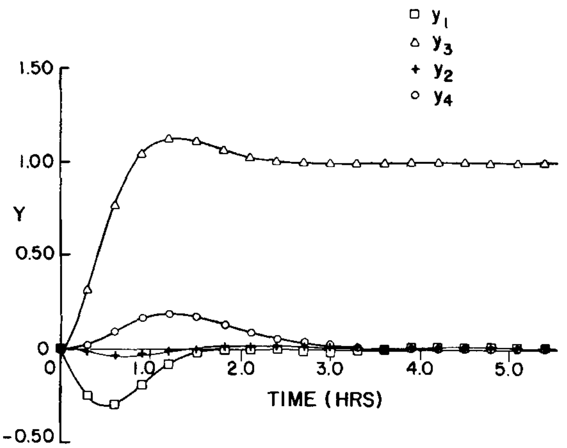

Figure 10a. Set point responses for the best 1-1 partitioning: $\{y_1, u_1\}, \{y_2, u_3\}, \{y_3, u_2\}, \{y_4, u_4\}$.

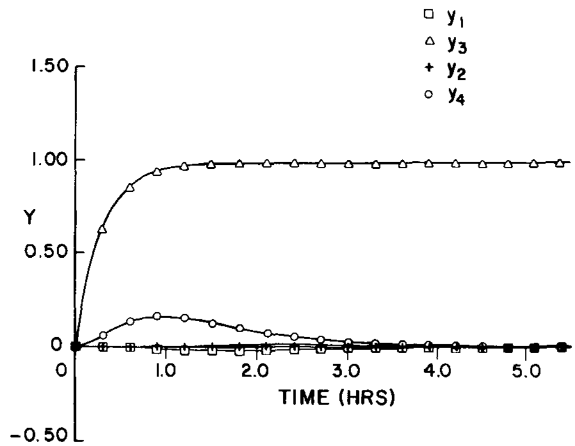

Figure 10c. Set point responses for the 2-2 partitioning: $\{y_1, y_3; u_1, u_2\}, \{y_2, y_4; u_3, u_4\}$.

	u_j			
y_i	1 + 2 + 3	1 + 2 + 4	1 + 3 + 4	2 + 3 + 4
1	0.999	1.039	0.531	0.429
2	1.001	0.100	0.929	0.971
3	1.091	0.830	0.555	0.525
4	−0.089	1.031	0.985	1.077

The only feasible alternative is $\{y_1, y_2, y_3; u_1, u_2, u_3\}$. The corresponding BRG_ℓ at $s = 0$ is given below:

Three-dimensional BRG_ℓ at $s = 0$

	u_j		
y_i	1	2	3
1	0.999	0.0	0.144
2	0.0	1.001	0.044
3	0.0	0.0	1.091

Checking the diagonal elements and the eigenvalues at all frequencies, this three-dimensional BRG turns out to be viable. When it is combined with the one-dimensional BRG corresponding to $\{y_4, u_4\}$, another acceptable partitioning develops:

Alternative 3. Block 1: $\{y_1, y_2, y_3; u_1, u_2, u_3\}$; Block 2: $\{y_4, u_4\}$.

By tightening the radii ϵ_1 and ϵ_2, one can identify this last alternative to be the most promising structure.

Dynamic simulations, Figure 10, verify all these results. The 1-1 SISO partitioning that is not recommended by the BRG analysis is the worst decentralized control structure. All three alternatives selected after the screening process show satisfactory and comparable performances. Similar results hold for other set point changes as well (Savage, 1984).

Acknowledgment

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Notation

$(BRG)_\ell$ = left block relative gain
$(BRG))_r$ = right block relative gain
F = feedback filter

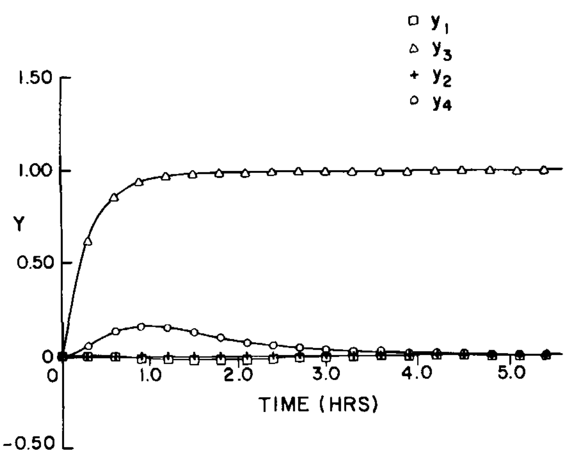

Figure 10b. Set point responses for the 2-1-1 partitioning: $\{y_1, y_3; u_1, u_2\}, \{y_2, u_3\}, \{y_4, u_4\}$.

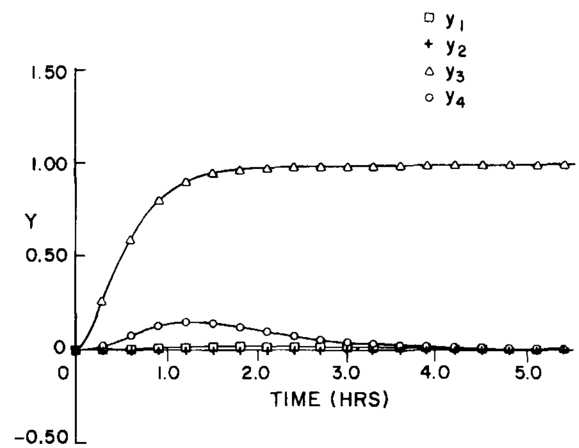

Figure 10d. Set point responses for the 3-1 partitioning: $\{y_1, y_2, y_3; u_1, u_2, u_3\}, \{y_4, u_4\}$.

G = plant transfer function matrix
G_{ii} = transfer function matrix for the ith subsystem
$\hat{G}_{ii}$ = noninvertible part of G_{ii}
$\tilde{G}_{ii}$ = invertible part of G_{ii}
G_{li} = model reference controller for the ith subsystem
$G_{11}, G_{12}, G_{21}, G_{22}$ = partitioned submatrices of G
I = identity matrix
K = controller matrix
m = dimension of BRG
N = number of subsystems
n = number of inputs and outputs
n_i = number of inputs and outputs in the ith subsystem
P = permutation matrix
R = set of real matrices
S = scaling matrix
s = Laplace variable
u = vector of n inputs
u_1 = vector of m inputs
u_2 = vector of $n - m$ inputs
u_j = jth input
y = vector of n outputs
y_1 = vector of m outputs
y_2 = vector of $n - m$ outputs
y_i = ith output

Greek letters

α = controller tuning parameter
λ = eigenvalue
ω = frequency

Superscripts

sp = set point
s = scaled

Special symbols

$\nexists$ = there does not exist

Appendix

Proof of Theorem 3

From theorem 1 we have

$$(BRG)'_\ell = P_{11} \cdot BRG_\ell \cdot P_{11}^T$$
$$(BRG)'_r = P_{21}^T \cdot BRG_r \cdot P_{21}$$

Then for m-dimensional BRG's we have

$$(BRG)'_\ell(i,i) = \sum_{k=1}^{m} P_{11}(i,k) \cdot (BRG_\ell \cdot P_{11}^T)(k,i)$$
$$= \sum_{k=1}^{m} P_{11}(i,k) \cdot \sum_{j=1}^{m} BRG_\ell(k,j) \cdot P_{11}^T(j,i)$$
$$= \sum_{k=1}^{m} P_{11}(i,k) \cdot \sum_{j=1}^{m} BRG_\ell(k,j) \cdot P_{11}(i,j)$$

Since P_{11} is a permutation matrix we have that for a given i there exists a unique $h = h(i)$

$$P_{11}(i,j) = \begin{matrix} 0 & \text{if} & j \neq h \\ 1 & \text{if} & j = h \end{matrix}$$

Then

$$(BRG)'_\ell(i,i) = \sum_{k=1}^{m} P_{11}(i,k) \cdot BRG_\ell(k,h) = BRG_\ell(h,h),$$

which is also a diagonal element. Similar arguments hold for $(BRG)'_r$ and the proof is complete.

Proof of Theorem 4

a) From theorem 2

$$(BRG)^s_\ell = S_{11}\, BRG_\ell \cdot S_{11}^{-1}$$
$$(BRG)^s_r = S_{21}^{-1}\, BRG_r \cdot S_{21}$$

Then

$$(BRG)^s_\ell(i,i) = \sum_{j=1}^{m} S_{11}(i,j)\,(BRG_\ell \cdot S_{11}^{-1})(j,i) \xrightarrow{S_{11}\text{ diagonal}}$$
$$(BRG)^s_\ell(i,i) = S_{11}(i,i) \sum_{h=1}^{m} BRG_\ell(i,h) \cdot S_{11}^{-1}(h,i) \xrightarrow{S_{11}^{-1}\text{ diagonal}}$$
$$(BRG)^s_\ell(i,i) = S_{11}(i,i) \cdot BRG_\ell(i,i) \cdot S_{11}^{-1}(i,i)$$
$$= BRG_\ell(i,i)$$

Similarly for $(BRG)^s_r$.

b) $$(BRG)^s_\ell(i,i) = [(G_{11}) \cdot (G^{-1})_{11}](i,i)$$
$$= \sum_{j=1}^{m} G_{11}(i,j) \cdot (G^{-1})_{11}(j,i)$$

Since G_{11} and $(G^{-1})_{11}$ consist of the first m inputs and outputs

$$G_{11}(i,j) = G(i,j)$$
$$(G^{-1})_{11}(j,i) = G^{-1}(j,i)$$

Thus

$$(BRG)_\ell(i,i) = \sum_{j=1}^{m} G(i,j) \cdot G^{-1}(j,i)$$
$$(BRG)_\ell(i,i) = \sum_{j=1}^{m} \text{RGA}(i,j)$$

Similarly

$$(BRG)_r(i,i) = [(G^{-1})_{11} \cdot (G_{11})](i,i)$$
$$= \sum_{j=1}^{m} (G^{-1})_{11}(i,j) \cdot (G_{11})(j,i)$$
$$= \sum_{j=1}^{m} G^{-1}(i,j) \cdot G(j,i)$$
$$= \sum_{j=1}^{m} G(j,i) G^{-1}(i,j)$$
$$= \sum_{j=1}^{m} \text{RGA}(j,i)$$

Proof of Theorem 5

a) From theorem 2

$$(BRG)^s_\ell = S_{11}\, BRG_\ell \cdot S_{11}^{-1}$$
$$(BRG)^s_r = S_{21}^{-1}\, BRG_r \cdot S_{21}$$

But then $(BRG)_\ell^s$ is similar to BRG_ℓ (S_{11} is a similarity transformation) and thus

$$\lambda_i(BRG)_\ell^s = \lambda_i(BRG)_\ell \forall i=1,\ldots,m$$

In the same way

$$\lambda_i(BRG)_r^s = \lambda_i(BRG)_r \forall i=1,\ldots,m$$

b) From theorem 1

$$(BRG)_\ell' = P_{11} \cdot BRG_\ell \cdot P_{11}^T$$
$$(BRG)_r' = P_{21}^T \cdot BRG_r \cdot P_{21}$$

Again since P_{11}, P_{21} are permutation matrices they are orthogonal. Thus

$P_{11}^T = P_{11}^{-1}$, $P_{21}^T = P_{21}^{-1}$ and $(BRG)_\ell'$, BRG_ℓ are similar.

Consequently

$$\lambda_i(BRG)_\ell' = \lambda_i(BRG)_\ell \forall i=1,\ldots,m.$$

In the same way $\lambda_i(BRG)_r' = \lambda_i(BRG)_r$.

c) It holds (see Kailath, 1980, Appendix) that

$$\lambda_i(A \cdot B) = \lambda_i(B \cdot A) \forall i=1,\ldots,\dim(A \cdot B)$$

when A, B are square.

Since

$$BRG_\ell = G_{11} \cdot (G^{-1})_{11}$$
$$BRG_r = (G^{-1})_{11} \cdot G_{11} \text{ and } G_{11}, (G^{-1})_{11} \text{ are square,}$$

it obviously holds that

$$\lambda_i(BRG)_\ell = \lambda_i(BRG)_r \forall i=1,\ldots,m$$

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